Phillips Curves, Expectations of Inflation and Optimal Unemployment Over Time

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This article is a study of the "optimal" fiscal control of aggregate demand. It presents a dynamic macroeconomic model from which is derived the optimal time-path of aggregate employment. Given this employment path and the initially expected rate of inflation, the time-path of the actual rate of inflation (positive or negative) can also be derived. If I am right about the dynamic elements, the problem of optimal demand is sufficiently difficult to justify some drastic simplifications in this first analysis: a closed, non-stochastic economy is postulated in which exogenous monetary policy immunizes investment against variations in capacity utilization in such a way as to keep potential capital intensity constant over time. But despite these limitations, I believe that the analysis introduces some important desiderata for national and international policy towards aggregate demand.

The principal ingredients of the model are the following: first, a sort of Phillips Curve in terms of the rate of price change, rather than wage change, that shifts one-for-one with variations in the expected rate of inflation; second, a dynamic mechanism by which the expected inflation rate adjusts gradually over time to the actual inflation rate; third, a social utility function that is the integral of the instantaneous "rate of utility" (possibly discounted) at each point in time now and in the future; last, a derived dependence (from underlying considerations of consumption and leisure) of the rate of utility at any time upon current "utilization" or employment—the decision variable under fiscal control—and upon the money rate of interest, hence, given the real rate of interest, upon the expected rate of inflation. An optimal utilization or employment path is one which maximizes the social utility integral subject to the adaptive expectations mechanism that governs the shifting of the Quasi-Phillips Curve.

The choice problem just sketched is dynamical: an optimal utilization policy by the government must weigh both the current benefits and the consequences for future utility possibilities of today's utilization decision. By contrast, the conventional approach to the employment-

[1] This article was written during my tenure of a Social Science Research Council Faculty Research Grant in the Spring of 1966 at the London School of Economics. I am very grateful to numerous economists there and at the Universities of Cambridge, Essex and York for their helpful comments on oral presentations. David Cass and Tjalling Koopmans of Yale University kindly scrutinized certain technical aspects of a preliminary and more extensive version of this article to which I shall make occasional reference: "Optimal Employment and Inflation Over Time", Cowles Foundation Discussion Paper No. 214 (August 1966). Any errors and other defects in this article are my responsibility.

inflation problem—if there is a conventional approach—is wholly statical.[1] I shall briefly describe that approach and show where I believe it goes wrong. Then I shall summarize the conclusions of the dynamical approach and attempt an intuitive explanation of them for those readers who do not wish to study the model in detail.

Visualize a diagram on which we represent the locus of unemployment-inflation combinations available to the government *when the expected rate of inflation equals zero* by a characteristically shaped Phillips Curve.[2] This curve is negatively sloped, strictly convex (bowed in toward the origin) and it intersects the horizontal axis at some unemployment ratio, say u^*, $0 < u^* < 1$. The quantity u^* measures the "equilibrium" unemployment ratio, for it is the unemployment rate at which the actual rate of inflation equals the expected rate of inflation so that the expected inflation rate remains unchanged. Now superimpose on to the diagram a family of social indifference curves, negatively sloped (at least in the positive quadrant) and strictly concave, and suppose that one of these indifference curves is tangent to the Phillips Curve at some unemployment ratio, say $\tilde{u}$, smaller than u^*. The quantity $\tilde{u}$ measures the (statical) optimum in the conventional approach. The inequality $\tilde{u} < u^*$ stems from the customary (though not unanimous) judgment that there is some reduction of unemployment below u^* that is worth the little inflation it entails.

But if the statical "optimum" is chosen, it is reasonable to suppose that the participants in product and labour markets will learn to expect inflation (and the concomitant money wage trend) and that, as a consequence of their rational, anticipatory behaviour, the Phillips Curve will gradually shift upward (in a uniform vertical displacement) by the full amount of the newly expected and previously actual rate of inflation. Now if the recalculated "optimal" unemployment ratio does not change in the face of the shift, greater inflation will result than before and the pattern will repeat as expectations are continually revised upwards; there will occur what is popularly called a "wage-price spiral" that is "explosive" or "hyper-inflationary" in character. It is more likely that the upward displacement of the Phillips Curve will cause the policy-makers to "take out" the loss in the form of an increase in the unemployment ratio as well as some increase in the rate of inflation. The rate of inflation will continue to increase as long as the unemployment ratio is smaller than u^*, so that the actual rate of inflation exceeds the expected rate with the consequence that the Phillips Curve is rising; but as the statically "optimal" $\tilde{u}$ approaches

[1] A recent example of the approach I have in mind is R. G. Lipsey, "Structural and Deficient-Demand Unemployment Reconsidered", in A. M. Ross (ed.), *Employment Policy and the Labor Market*, Berkeley, 1965. See also A. M. Okun, "The Role of Aggregate Demand in Alleviating Unemployment", in *Unemployment in a Prosperous Economy*, A Report of the Princeton Manpower Symposium, May 13-14, 1965, Princeton, N. J., pp. 67-81.

[2] The classic reference of course is W. A. Phillips, "The Relation Between Unemployment and the Rate of Change of Money Wage Rates in the United Kingdom, 1861-1957", *Economica*, vol XXV (1958), pp. 283-99.

u^*, a stationary equilibrium will be asymptotically reached in which $\hat{n} = u^*$ and there is equality between the expected and actual rates of inflation. Even though a state of steady inflation is eventually achieved, it is likely to be a very high rate of inflation—much higher than the policy-makers myopically bargained for. Thus the conventional approach goes wrong in implicitly discounting future utilities infinitely heavily.[1] (This is not the only amendment to the conventional approach that I shall make.)

The dynamical approach recognizes that any optimal time-path of the unemployment ratio must approach the steady-state equilibrium level, u^*; perpetual maintenance of the unemployment ratio below that level (perpetual over-employment) would spell eventual hyper-inflation and ultimately barter, while perpetual maintenance of unemployment above that level (perpetual under-employment) would be wasteful of resources. The policy trade-off is not a timeless one between permanently high unemployment and permanently high inflation but a dynamic one: a more inflationary policy permits a transitory increase of the employment level *in the present* at the expense of a (permanently) higher inflation and higher interest rates in the *future* steady state. Optimal aggregate demand therefore depends upon society's time preference.

If there is no time discounting of future utilities, future considerations dominate and society should aim to achieve asymptotically the best of all possible steady states, namely the one in which the (actual and expected) inflation rate is low enough, and hence the money interest rate (the cost of holding money) is low enough, to satiate the transaction demand for liquidity by eliminating private efforts to economize on cash balances. If that steady state is not realizable immediately at the equilibrium unemployment ratio, because the initially expected rate of inflation is too high, society should accept under-employment in order to drive down the expected rate of inflation to the requisite point and thus permit an asymptotic approach to the desired steady state. If society has a positive discount rate, it will pay to trade off an ultimate shortfall of liquidity in the future steady state—to accept an ultimately higher rate of inflation and hence a higher cost of holding money—for higher employment in the present; the steady state chosen will be more inflationary the greater the discount rate. If that ultimately desired steady state does not now obtain at equilibrium unemployment because the initially expected inflation rate is *too high*, under-employment must still be accepted in order to drive down the expected inflation rate. But, symmetrically, if the initially expected rate of inflation is *below* the ultimately tolerated rate of inflation, *over-employment* is optimal to drive up the expected inflation rate. (In both cases, unemployment gradually approaches the equilibrium level as the expected inflation rate approaches the ultimately desired level.) Clearly, over-employment is more likely to be appropriate the greater is the discount rate; optimal employment in the present is an increasing function of the discount rate. Thus optimal employment policy in this dynamic model depends to an important extent upon time preference.[1]

Now for the construction, defence and analysis of the model. In this publication I confine myself to the simplest version with an infinite decision-making horizon, a smooth utility function and an equilibrium "utilization" ratio that is independent of the rate of inflation (as in the above discussion).

I. Possibilities and Preferences

In this part the model is developed and the optimization problem stated. The solution will be discussed in Parts II and III.

A. *The "virtual" golden age, utilization and interest.* To make the money rate of interest a stationary function of employment or utilization, to make only consumption, not investment, vary with utilization—both in order to simplify preferences—and to make the marginal productivity of labour rise at the same constant proportionate rate for every employment or utilization ratio—in order that the notion of a stationary family of Phillips Curves in terms of prices have greater plausibility—I postulate that the economy, thanks to a suitably chosen monetary policy and to the nature of population growth and technological progress, is undergoing "virtual" golden-age growth. By this I mean that actual golden-age growth *would be observed* in the economy if the employment-labour force ratio or utilization ratio were constant. (Golden-age growth is said to occur when *all* variables change exponentially, so that investment, consumption and output grow at the same rate which may exceed the rate of increase of labour.)

[1] Of course, my criticism is founded also upon the postulated "instability" of the Phillips Curve. In fact, a situation of sustained "over-employment"—more precisely unemployment less than u^* by a non-vanishing amount— has been supposed to produce an explosive spiral through its effects upon the Phillips Curve. On my assumptions, the only *steady-state* Phillips Curve is a vertical line intersecting the horizontal axis at u^*. Now some econometric work over the past ten years might suggest that, especially on a fairly aggregative level, the Phillips Curve is a tolerably stable empirical relationship. But these studies probably estimate some average of different Phillips Curves, corresponding to different expected rates of inflation and of wage change which have varied only over a small range. Further, some writers have found the actual rate of inflation to have a weak influence on wage change and this may be explained by the view that the actual rate of inflation is a proxy, but a very poor one, for the expected rate of price or wage change. See, with reference to British data, R. G. Lipsey, "The Relation Between Unemployment and the Rate of Change of Money Wage Rates in the United Kingdom, 1862-1957: A Further Analysis", *Economica*, vol. XXVII (1960), pp. 1-31; and, with reference to American data, G. L. Perry, "The Determinants of Wage Rate Changes and the Inflation-Unemployment Trade-off in the United States," *Review of Economic Studies*, vol. 31 (1964), pp. 287-308.

[1] If the Phillips Curve shifts upward with a one point increase of the expected inflation rate by less than one point, then the steady-state Phillips Curve will be negatively sloped. But it will be steeper than the non-steady-state Phillips Curves which is all that is required to justify a dynamical analysis and to make the discount rate important. It is true, however, that the criticism of the statical approach loses more of its force and the discount rate is less important the less steep is the steady-state curve in relation to the non-steady-state curves. A case of a negatively sloped steady-state Phillips Curve is analysed in my preliminary paper, "Optimal Employment and Inflation Over Time," *op. cit.*

To generate virtual golden-age growth I suppose that the homogeneous labour force (or competitive supply of labour) is homogeneous of degree one in population and homogeneous of degree zero in the real wage, disposable real income per head and real wealth per head.[1] Hence, whenever the latter three variables are changing equiproportionately, the labour supply will grow at the population growth rate, say γ. More general assumptions are apt to impair the feasibility of golden-age growth.

As for production, let us think in terms of an aggregate production function which exhibits constant returns to scale in capital and employment with technical progress, if any, entering in a purely labour-augmenting way, so that output is a linear homogeneous function of capital and augmented employment (or employment measured in "efficiency units"). Suppose further that the proportionate rate of labour augmentation is a non-negative constant $\lambda \geq 0$. Then augmented labour supply will grow exponentially at the "natural" rate, $\gamma + \lambda > 0$, whenever the real wage rate, disposable real income per capita and real per capita wealth grow in the same proportion.

As for capital, we require that the capital stock grow exponentially at the rate $\gamma + \lambda$. Then output will grow exponentially, as will investment and hence consumption, at the rate $\gamma + \lambda$ for any constant augmented employment-capital ratio—which I shall call the *utilization ratio*. This implies that the government, by monetary actions I shall assume, always brings about the right level of (exponentially growing) investment necessary for exponential growth of capital at the natural rate.

On these assumptions there is virtual golden-age growth: at any *constant* utilization ratio, output, investment, consumption, capital, augmented employment and, under marginal productivity pricing, real profits and real wages will all grow exponentially at the natural rate, while the marginal and average product of labour and, under marginal productivity pricing, the real wage rate, real income per capita and real wealth per capita will all grow at the rate λ. Disposable real income per head will also grow at rate λ on plausible assumptions (e.g., a constant average propensity to consume) such that the taxes per head necessary for the exponential growth of consumption per head also grow at rate λ. Thus the labour supply will grow at rate γ, like population and employment. The marginal product of capital and the equilibrium competitive real interest rate will be constant over time. (If the augmented employment-capital ratio is changing over time, most of these variables will not be growing exponentially; it is only population, labour augmentation, capital and investment that grow exponentially, come what may.)

[1] Taxes will be lump-sum. Labour supply is supposed independent of the real and money rates of interest. I neglect the difference between wealth and capital, i.e., the government debt. This is acceptable if the wealth-capital ratio is constant over time. While this will not occur in my model, that ratio will become asymptotic as any golden-age path is approached. I suggest therefore that the error is small enough to be neglected safely.

While the monetary authority (the Bank) is postulated to guide investment along its programmed path, the fiscal authority has control over consumption demand and hence, given the programmed investment demand, aggregate demand and employment. Since employment is the decision variable in the present problem, fiscal devices are the policy instruments by which consumption demand and thus employment are controlled. I postulate unrealistically that the Fisc levies "lump-sum" taxes (taxes having no substitution effects) on households for this purpose.

The monetary instruments by which the Bank keeps investment on its programmed path are assumed to be devices like open-market operations which operate through the rate of interest or directly upon the demand for capital. The Bank must be alert therefore to adjust interest rates in the face of changes in aggregate demand or utilization engineered by the Fisc. If the real interest rate equals or is closely tied to the marginal productivity of capital, then clearly the real rate of interest will be higher the greater is the utilization ratio, since investment is to be kept on the exponential path appropriate to virtual golden-age growth.[1] Now to the details.

The real rate of interest is the money rate of interest minus the expected rate of inflation. I assume here that expectations of the current price trend are held unanimously and certainly by the public (but not necessarily by the policy-makers who, from this point of view, lead an unreal existence). If we let i denote the money rate of interest and let r denote the real rate of interest, we obtain

(1) $i = r - x, \quad 0 \leq i < i_0,$

where x is the expected rate of algebraic *deflation*. Thus $-x$ is the expected rate of inflation.[2] Equation (1) says, therefore, that as x becomes algebraically small, i.e., as inflation becomes expected, the money rate of interest becomes high, given the real rate of interest; for given the physical or real yield on capital, the prospects of high nominal capital gains on physical assets (and hence on equities) produced by the

[1] We do usually observe that interest rates are relatively high in "good times", but evidently they are not sufficiently high or high soon enough to prevent pro-cyclical variations of investment expenditures. Possibly the reason is that business fluctuations are too sharp and imperfectly foreseen to permit the monetary authorities to stabilize investment. But if fiscal weapons were used effectively to control consumption demand, as they are assumed to be in this article, then the Bank's job of controlling investment would be much facilitated. It must be admitted, however, that the whole question of optimal fiscal and monetary policy in the presence of exogenous stochastic shocks and policy lags is beyond the scope of this article.

It should also be mentioned that the exclusive assignment of investment control to the monetary authority is inessential to this article. Indeed, it might be more realistic to suppose that investment was controllable in the desired manner through fiscal weapons. But then one could not identify the real rate of interest even loosely with the pre-tax marginal product of capital so there would be no simple interpretation of the shape of the $r(y)$ function in equation (2).

[2] I know that I owe the reader an apology for inflicting this notation on him. I have chosen to work in terms of expected deflation in order to emphasize its resemblance to capital in the well-known problem of optimal saving, a problem having some similarity to the present one.

expectation of inflation will induce people to ask a high interest rate on the lending of money, while borrowers will be prepared to pay a high rate since the loan will be expected to be repaid in money of a lower purchasing power.

Since no one will lend money at a negative money rate of interest when he can hold money without physical cost, the money rate of interest must be non-negative. Further, it is assumed that there is a constant, i_b, to be called the "barter point", such that at any money interest rate equal to or in excess of it money ceases to be held so that the monetary system breaks down; this is because such a high money rate of interest imposes excessive opportunity costs on the holding of non-interest-bearing money instead of earning assets like bonds and capital.

As indicated previously, the real rate of interest will be taken to be an increasing function of the utilization ratio, denoted by y:

$$(2) \quad r = r(y),\ r(y) > 0,\ r'(y) > 0,\ r''(y) \geqslant 0, \qquad 0 < \mu \leqslant y \leqslant \bar{y} < \infty.$$

Consider the bounds on the utilization ratio. If positive employment is required for positive output then, by virtue of diminishing marginal productivity of labour, there is some small utilization ratio, denoted by μ, such that output will be only large enough to permit production of the programmed investment, leaving no employed resources for the production of consumption goods. Since negative consumption is not feasible, no value of y less than μ is feasible. The value μ is a constant by implication of the previous postulates. In the other direction, there is clearly, at any time, an upper bound on (augmented) employment arising from the supply of labour function and the size of population. This explains the upper bound $\bar{y}$ which, quite plausibly in view of the previous assumptions, is taken to be a constant.

Consider now the $r(y)$ function itself in the feasible range of the utilization ratio. The postulate that $r(y) > 0$ for all feasible y is perhaps not unreasonable; it could be relaxed. The curvature of $r(y)$ is of greater importance. (The later Figure 2 gives a picture of this function.) On the view that r is equal to the marginal product of capital, one is in some difficulty, for there are innumerable production functions that make the marginal product of capital a strictly *concave* (increasing) function of the labour-capital ratio, e.g., the Cobb-Douglas. Fortunately, I do not really require convexity of $r(y)$; $r''(y) \geqslant 0$ is overly strong for my purpose which, it will later be clear, is the concavity of U in y in (8). (Even the latter concavity could probably be dispensed with by one more expert than the present author in dynamic control theory, though probably the solutions would be somewhat affected.) I shall later indicate the minimum requirement on $r''(y)$. Moreover, there are countless production functions which make $r''(y) \geqslant 0$; for example, any production function which makes the marginal-product-of-labour curve linear or strictly convex in labour (which is not customary in textbooks) will suffice and even some concavity is consistent with (2).

Finally, a word about the use of the ratio of augmented labour to capital as a strategic variable in the model. Since capital is growing like $e^{(\gamma+\lambda)t}$ while employment is multiplied by $e^{\lambda t}$ to obtain augmented employment, it can be seen that, if N denotes employment and K denotes capital, then, with suitable choice of units,

$$y = \frac{e^{\lambda t} N}{K} = \frac{e^{\lambda t} N}{e^{(\gamma+\lambda)t}} = \frac{N}{e^{\gamma t}}.$$

Hence the definition of the utilization ratio used here does not imply a neo-classical model with aggregate "capital" in the background. Only neo-classical *properties* like diminishing marginal productivities need to be postulated and these are much more general than the neo-classical model. The previous relation shows that we could as well define the utilization ratio as the employment-population ratio (since population is growing like $e^{\gamma t}$) which, in the present model, is a linear transformation of the augmented employment-capital ratio. Thus the utilization ratio here measures not only the intensity with which the capital stock is utilized (the number of augmented men working with a unit of capital) but also the utilization of the population in productive employment.

B. *Inflation, utilization and expectations.* I am going to postulate that the rate of inflation depends upon the utilization ratio and upon the expected rate of inflation. In particular, the rate of inflation is an increasing, strictly convex function of the utilization ratio. When the expected rate of inflation is zero, the rate of inflation will be zero when the utilization ratio equals some constant y^* between μ and $\bar{y}$, will be positive for any greater utilization ratio and negative for any smaller utilization ratio. As $\bar{y}$ is approached, the rate of inflation approaches infinity. Finally, every increase of the expected rate of inflation by one point will increase by one point the actual rate of inflation associated with any given utilization ratio. Remembering that $-x$ is the expected rate of inflation, one therefore may write

$$(3) \quad \begin{array}{l} \dot{p}/p = f(y) - x, \qquad \mu \leqslant y \leqslant \bar{y}, \\ f'(y) > 0,\ f''(y) > 0,\ f(\bar{y}) = \infty,\ f(y^*) = 0, \quad \mu < y^* < \bar{y}, \end{array}$$

where p is the price level and $\dot{p}$ its absolute time-rate of change so that $\dot{p}/p$ is the rate of inflation. Thus we must add the expected rate of inflation to the function $f(y)$ to obtain the actual rate of inflation. For every x we have a Quasi-Phillips Curve relation between $\dot{p}/p$ and y. The relationship is pictured in Figure 1.

I believe there can be no real question that, if the somewhat Phillipsian notion of the $f(y)$ function is accepted, the expected rate of inflation must be added to it as in (3) if, as assumed, the supply of labour is independent of the real and money rates of interest and hence independent of the expected rate of inflation. If the matter were otherwise, every steady state of fully anticipated inflation would be associated with different "levels" of output, employment and the real wage. Note that no assumption of any kind concerning the formation of expectations

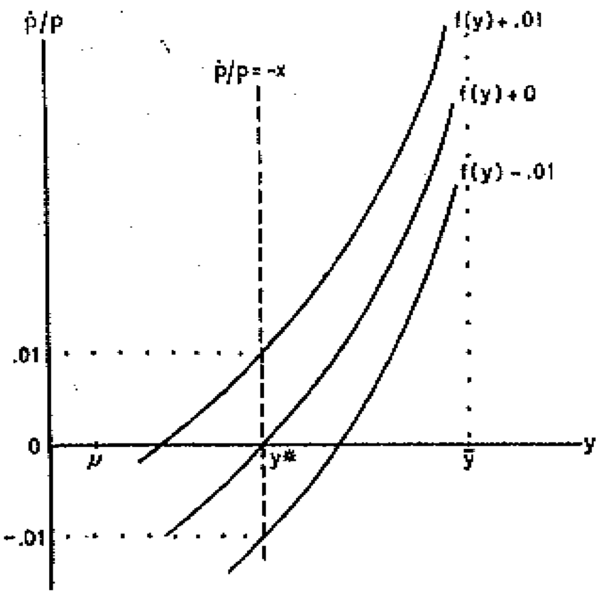

FIGURE 1. QUASI-PHILLIPS CURVES FOR $-x = \cdot 01, 0, -\cdot 01$.

has yet been made here; no assumption of perfect foresight or the like is implied in the formulation of this inflation function.

The concept of the function $f(y)$ is more vulnerable to criticism. From the usual Phillips Curve standpoint, we have to regard the utilization ratio as a proxy for the ratio of employment to labour supply and to neglect rising marginal cost. And of course the simple Phillips Curve itself is recognised to be an inadequate description of wage behaviour.

Looking at Figure 1 or equation (3) we see that y^* can be regarded as the *equilibrium* utilization ratio, for at $y = y^*$ (and only there) the actual rate of inflation will equal the expected rate of inflation. Mathematically, $\dot{p}/p = -x$ at $y = y^*$ since $f(y^*) = 0$. The diagram likewise shows that all the points on the vertical dashed line intersecting y^* are equilibrium points. Without intending normative significance, we may refer to $y > y^*$ as "over-utilization" and refer to $y < y^*$ as "under-utilization", merely from the point of view of equilibrium.

When there is over-utilization, the actual rate of inflation exceeds the expected rate, and *vice versa* when there is under-utilization. In either of these situations there will presumably be an adjustment of the expected rate in inflation. I shall adopt the mechanism of "adaptive expectations"

first used in this context by Phillip Cagan.[1] The (algebraic) absolute time-rate of increase of the expected rate of inflation will be supposed to be an increasing function of the (algebraic) excess of the actual rate of inflation over the expected rate, being equal to zero when the latter excess equals zero. Symbolically, if $(\dot{p}/p)^e$ denotes the expected rate of *inflation*, the postulate is

$$\frac{d}{dt}\left(\frac{\dot{p}}{p}\right)^e = a\left[\frac{\dot{p}}{p} - \left(\frac{\dot{p}}{p}\right)^e\right]$$

or, in terms of the expected rate of *deflation*,

$$-\dot{x} = a\left(\frac{\dot{p}}{p} + x\right),$$

(4)
$$a(0) = 0,\ a'(\ ) > 0,\ a''(\ ) > \frac{-a'(\ )f''(\ )}{f'(\ )f'(\ )}.$$

Concerning the curvature of the function $a(\dot{p}/p + x)$, it might be thought to be linear or it might be conjectured to be strictly convex for positive $\frac{\dot{p}}{p} + x$ and strictly concave for negative $\frac{\dot{p}}{p} + x$. All I am requiring is that the function not be "too concave" in the feasible range of y; in particular, it must not be more concave then the f function is convex, loosely speaking.

Substitution of (3) into (4) yields $-\dot{x} = a[f(y) - x + x] = a[f(y)]$. If we let $G(y)$ denote $-a[f(y)]$, then, by virtue of (3) and (4) we may write

(5)
$$x = G(y), \quad \mu \leqslant y \leqslant \bar{y},$$
$$G(y^*) = 0, \quad G'(y) < 0, \quad G''(y) < 0.$$

Thus, when $y = y^*$, the actual and expected inflation rates are equal so that there is no change in the expected rate of inflation. When $y > y^*$, so that the actual inflation rate exceeds the expected rate, the expected rate of inflation will be rising or, equivalently, the expected rate of *deflation* will be falling. The opposite results hold when $y < y^*$. Note that as y is increased, the rate at which the expected rate of *inflation* is increasing over time will increase with y at an increasing rate.

In order to determine the path of x over time as a function of the chosen y path, we need to know the (initial) x at time zero, $x(0)$, which we take to be a datum:

(6) $x(0) = x_o$.

We have to consider the admissible values of x_o in view of the upper and lower bounds on the money interest rate given in (1). First, for our analytical problem to be interesting, we require that x_o not be so algebraically small—that the initially expected *inflation* rate not be so great—that no feasible y decision by the Fisc can save the monetary system from breaking down in the first instant; that is, x_o must be sufficiently large algebraically that $i = r(y) - x_o < i_b$ for sufficiently

[1] P. Cagan, "The Monetary Dynamics of Hyperinflation," in M. Friedman (ed.), *Studies in the Quantity Theory of Money*, Chicago, 1956, pp. 25-117.

small $y \geq \mu$. Hence we require that $r(\mu) - x_o < i_b$ (or, in later notation, $x_o > x_b(\mu)$).

As for the non-negativity of the money interest rate, by analogous reasoning I should require only that x_o not be so large—that the initially expected deflation rate not be so great—that there is no y that will permit the Bank to make the real rate of interest low enough to induce the programmed volume of investment; that is, x_o must be sufficiently small that $i = r(y) - x_o \geq 0$ for sufficiently large $y < \bar{y}$, hence that $r(\bar{y}) - x_o \geq 0$. But I have to confess that I do not take seriously the non-negativity constraint in my analysis. To justify this neglect I want somewhat stronger assumptions that will prevent the constraint from becoming binding when an optimal policy is followed. The constraint will not be binding initially if $r(\mu) - x_o \geq 0$, since the chosen y must be at least as great as μ. If, further, we postulate that $r(\mu) - \hat{x}(y^*) \geq 0$, where $\hat{x}(y^*)$ is a "satiation" concept later defined, then the constraint will not be binding in the future either, for our solution will be seen to imply that the optimal $x(t) \leq \max[x_o, \hat{x}(y^*)]$ for all t. I believe these conditions are fairly innocuous (as well as over-strong) and that it is wise not to complicate the problem at this stage by serious consideration of the non-negativity constraint.

C. *Utilization, liquidity and utility.* The problem of the Fisc is to choose a path $y(t)$, $t \geq 0$, or, equivalently, a policy function, $y(x, \ldots)$ subject to (5), (6) and the information in (1), (2) and (3). For this the Fisc requires preferences. I shall follow Frank Ramsey in adopting a "social utility function" that is the integral over time of the possibly discounted instantaneous "rate of utility".[1]

On what variables should the (undiscounted) rate of utility, U, at any time t be taken to depend? I am going to suppose that the only two basic desiderata are consumption and leisure. On this ground I write the twice-differentiable function

(7) $U = \varphi(i, y) = \varphi[r(y) - x, y]$

where

(a) $\varphi_2 > 0$ for $y < y^0$, $y^* < y^0 < \bar{y}$,
$\varphi_2 < 0$ for $y > y^0$,
where $\varphi_2(i, y^0) = 0$ for all i, y^0 a constant.
$\varphi_{22} < 0$ for all y.
$\lim_{y \to \mu} \varphi = -\infty$, $\lim_{y \to \bar{y}} \varphi = -\infty$.

(b) $\varphi_1 = \varphi_{11} = \varphi_{12} = 0$ for $i \leq \bar{i}$, $0 \leq i < i_b$,
$\varphi_1 < 0$, $\varphi_{11} < 0$, $\varphi_{21} = \varphi_{12} \leq 0$ for $i > \bar{i}$,
where $\varphi_1(\bar{i}, y) = 0$ for all y, $\bar{i}$ a constant.
$\lim_{i \to i_b} \varphi = -\infty$.

[1] F. P. Ramsey, "A Mathematical Theory of Saving," *Economic Journal*, vol. 38 (1928), pp. 543-59. For a discussion in a different context of the axiomatic basis for such a utility function, see T. C. Koopmans, "Stationary Ordinal Utility and Impatience", *Econometrica*, vol. 28 (1960), pp. 287-309.

It should be noted that the function φ is taken to be determined up to a linear transformation so that the assumptions on the signs of the second partial derivatives are meaningful. Figure 2 shows the contours of constant U.[1] Now the explanation.

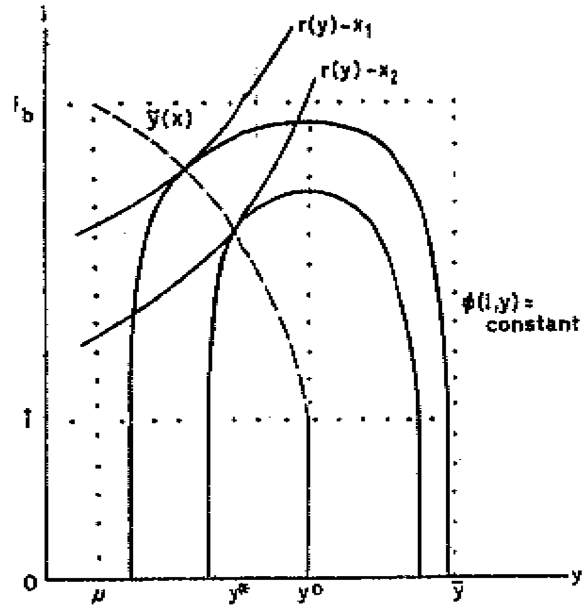

FIGURE 2: CONTOURS OF CONSTANT $\phi(i,y)$, THE INTEREST RATE FUNCTION AND THE FUNCTION $\bar{\bar{y}}(x)$.

Consider first the dependence of the rate of utility upon utilization *for a fixed money rate of interest.* That is, consider (7a). Clearly, as y is increased, there will be more output, assuming always positive marginal productivity of labour, so that, given exogenous investment, there will be more consumption. In addition, there will be a reduction of involuntary unemployment, at least in a certain range. But, on the other hand, there will also be a reduction of leisure. Further, a discrepancy between y and y^* implies the failure of expectations to be realized.

[1] The assumptions in (7) guarantee strictly diminishing marginal rate of substitution above $\bar{i}$ and to the left of y^*. But for convexity to the right of y^0 we require that φ_{11} not be "too negative." Fortunately the contours are of no interest to the right of y^0 so we need not bother to place a lower bound on φ_{21}.

which suggests that people will have wished they had made different decisions.[1]

To make order out of this tangle of conflicting influences on the utility rate, I suggest the following view. Suppose for the moment that there were a perfect homogeneous national labour market. Then y^* would be the market-clearing utilization ratio at which the gain from a little more income (or consumption) was just outweighed by the loss of leisure necessary to produce it; thus the utility peak would be at y^*. Since consumption is strictly concave in y while effort increases linearly with y, we would expect the curve to be strictly concave everywhere, i.e., dome-shaped. Moreover, as y approaches μ, so zero consumption is approached, the rate of utility can reasonably be supposed to go to minus infinity; similarly, as y approaches $\bar{y}$, it is perhaps natural to suppose that the rate of utility again goes to minus infinity (although nothing in the solution hinges on this strong assumption). In such a world, what permits the Fisc to coax employment in excess of y^* is the failure of people to predict the magnitude of the inflation; in this world, some real normative significance attaches to "over-utilization" or "over-employment".

But in the real world, where there are countless imperfections and immobilities among heterogeneous sub-markets for different skills of labour in different industries, an additional consideration is operative. In such a world, there is substantial involuntary unemployment in some (presumably not all) sectors of the economy and among certain skill categories of labour even in utilization equilibrium; the point y^* is characterized by a balance between excess demand in some sectors and excess supply in others. In view of this and the social undesirability (*ceteris paribus*) of involuntary unemployment, I have supposed in (7) that the dome-shaped utility curve reaches a peak at some constant y^0 greater than y^* but less than $\bar{y}$; but the rate of utility does decline with y beyond this point as the involuntary over-employment in some labour markets and other misallocations by individuals (due to their failure to expect the resulting inflation) become increasingly weighty.[2] I shall indicate later the effect of making $y^0 = y^*$ contrary to my postulate. Note that y^0 is a constant independent of the money interest rate; this simplifying assumption seems advisable for consistency with the earlier postulate that the supply of labour is independent of the money interest rate.

I have discussed (7a)—that is, the profile of φ against utilization for a given money rate of interest. (A diagram of the relation between U and y *for a given* x will be shown later.) Consider now the dependence of the

[1] With aggregate investment being fixed, people cannot save too much or too little in the aggregate. But they can work too much or too little as a consequence of incorrect expectations.

[2] In polling people to determine y^0 the Fisc does not reveal to people that the level of the money rate of interest depends upon their social choice of y; y^0 is, like y^* earlier, a utility peak at any *fixed* money interest rate. With regard to the y^0 peak, labour turnover and perhaps labour hoarding are also relevant.

rate of utility on the money interest rate *for a given utilization ratio*. The money rate of interest measures the opportunity cost of holding money in preference to earning assets since, in the absence of own-interest on money, the money interest rate measures the spread between the yield on earning assets and the yield on money. After a point, an increase of the money interest rate increases incentives to economize on money for transactions purposes by means of frequent trips to banks and the like. I shall suppose for simplicity that these time-consuming efforts fall on leisure rather than on labour supply as indicated earlier. As the money rate of interest approaches the "barter point", i_b, these activities become so onerous that money ceases to be held and the monetary system breaks down. At a sufficiently small (but positive) money interest rate, $\bar{i}$, or at any smaller interest rate, incentives to economize are weak enough to permit a state of "full liquidity" in which all transactions balances are held in the form of money.[1]

Thus, concerning the relation between φ and i for given y, I suppose that the curve is flat in the full-liquidity range, $0 \leq i \leq \bar{i}$, negatively sloped and strictly concave for greater i and that the curve approaches minus infinity as i approaches i_b. I do not care how close $\bar{i}$ and i_b are to one another as long as they are separated. By making the curve go to minus infinity I insure that the optimal policy is not one producing the breakdown of the monetary system. I have now explained (7b) except for the condition that $\varphi_{21} = \varphi_{12} \leq 0$. This means that an increase of the money interest rate (outside the full-liquidity range) decreases or leaves unchanged the marginal utility of utilization; this seems reasonable since both an increase of i and of y imply a reduction of leisure, making leisure more or at least not less valuable at the margin.

It is clear from Figure 2 that, given the dependence of the interest rate on utilization, neither the value of y such that $i = \bar{i}$ (full liquidity) nor $y = y^0$ is generally a statical optimum, i.e., gives the maximum current rate of utility. The decision to make $i = \bar{i}$ may cost too much in terms of under-utilization while the decision $y = y^0$ may entail too high an interest rate. As Figure 2 shows, the static optimum is at $\hat{y}$ which is an increasing function of x up to y^0. If the Fisc sought to maximize the current rate of utility (which it is not optimal to do), it would (except in the case of a no-tangency, full-liquidity solution) equate the marginal rate of substitution, $-\varphi_1/\varphi_2$, to the slope of the i-function, $r'(y)$, taking out any gain from a downward shift of the i-function—of an increase of x—in the form of greater y and smaller i; for all x greater than or equal

[1] A formal analysis of interest and "full liquidity" is contained in my paper, "Anticipated Inflation and Economic Welfare", *Journal of Political Economy*, vol. 73 (1966), pp. 1-13. That paper deliberately neglects the steps necessary to establish the desired expected inflation rate in the particular case where, as here, no interest can be paid on money; it is entirely comparative statics, unlike the present paper. Incidentally, it is assumed there too that the lost time from economizing on money is "taken out" in the form of a leisure reduction rather than a labour-supply reduction (in order to facilitate diagrammatic analysis). The present paper does not assume knowledge of that paper.

to some large x, say $\hat{x}(\tilde{y})$, $\tilde{y}$ is identical of y^0 and $i \leqslant \hat{i}$ (full liquidity) as the diagram shows.

We need now to describe the rate of utility as a function of x and y, i.e., taking both the direct effect and the indirect effect through i, given x, of a change of y. From (2) and (7) we obtain

(8) $U = U(x,y), \quad 0 \leqslant r(y) - x < i_b, \quad \mu \leqslant y \leqslant \tilde{y}$

$U_y = \varphi_1 r'(y) + \varphi_2 > 0$ for $y < \bar{y}(x)$

$U_y < 0$ for $y > \bar{y}(x), \quad \mu < \bar{y}(x) \leqslant y^0,$

where $U_y(x,\bar{y}) = \varphi_1[r(\bar{y}) - x]\, r'(\bar{y}) + \varphi_2[r(\bar{y}) - x, \bar{y}] = 0.$

$U_{yy} = \varphi_{11} r'(y) r'(y) + 2\varphi_{21} r'(y) + \varphi_{22} + \varphi_1 r''(y) < 0$ (for all y).

$U_{yx} = -\varphi_{11} r'(y) - \varphi_{21} \left\{\geqslant\right\} 0$ as $x \left\{\begin{matrix}\leqslant\\ \geqslant\end{matrix}\right\} \hat{x}(y)$ or $i \left\{\begin{matrix}\geqslant\\ \leqslant\end{matrix}\right\} \hat{i}.$

$\bar{y}'(x) = -U_{yx}/U_{yy} \geqslant 0.$

$\lim_{y \to \mu} U = -\infty, \quad \lim_{y \to \min[y_b(x), \tilde{y}]} U = -\infty$

where $r(y_b) - x = i_b, \; y_b'(x) > 0\cdot$

(b) $U_x = U_{xx} = U_{xy} = 0$ for $x \geqslant \hat{x}(y)$

$U_x = -\varphi_1 > 0, \; U_{xx} = \varphi_{11} < 0, \; U_{xy} = -\varphi_{11} r'(y) - \varphi_{12} < 0$

for $x < \hat{x}(y)$,

where $r(y) - \hat{x} = \hat{i}, \; \hat{x}'(y) > 0.$

$\lim_{x \to x_b(y)} U = -\infty$ where $r(y) - x_b = i_b, \; x_b'(y) > 0.$

(c) $U(\hat{x}, y^*) = \varphi(\hat{i}, y^*) = \hat{U}.$

Let us first interpret the new notation before looking at the diagrams. The function $\bar{y}(x)$ has already been explained; it denotes the y at which the rate of utility is at a maximum with respect to y, taking into account the influence of y upon i, given x. The quantity y_b, also an increasing function of x, is that value of y which, given x, is just large enough to cause a breakdown of the monetary system by virtue of its causing $i = i_b$ through the $r(y)$ function; of course, x may be large enough to make $y_b > \tilde{y}$ in which case y_b is irrelevant; it will be relevant if x is so negative that the economy is teetering on the edge of barter. The quantity $\hat{x}$, which is an increasing function of y, is that value of x just sufficiently great, given y, to permit full liquidity, to permit $i = \hat{i}$; since an increase of y entails a higher r, i.e., $r'(y) > 0$, we shall need greater x to maintain $i = \hat{i}$ the higher is y; of course, any $x > \hat{x}(y)$ is also consistent with full liquidity, as $\hat{x}$ is the *minimum* x consistent with full liquidity. The quantity x_b, which is certainly negative even for large y, is that value of x so small algebraically that, given y, $i = i_b$ so that the monetary system breaks down; since $r'(y) > 0$, an increase of y causes an algebraic increase of x_b for we then need a smaller expected *inflation* rate to save the economy from barter. Finally, as a matter of notation, $\hat{U}$ denotes the rate of utility at equilibrium utilization and full liquidity, i.e., at $y = y^*$ and $x \geqslant \hat{x}(y^*)$; $\hat{U}$ is the maximum *sustainable* rate of utility.

Figure 3 illustrates the dependence of the utility rate on y, allowing for the interest effect of utilization, for two particular values of x: first, $x = \hat{x}(y^*)$ so that there will be full liquidity at $y = y^*$ (and at smaller y); second, $x = x_1 < \hat{x}(y^*)$, i.e., at a smaller x. I have supposed for the sake of definiteness that x_1 is so small—very negative—that when $x = x_1$ full liquidity is not realizable even at very small y so that the two curves never coincide; and that $x_b(y^*) < x_1$ so that the right-hand asymptotic lies to the right of y^*.

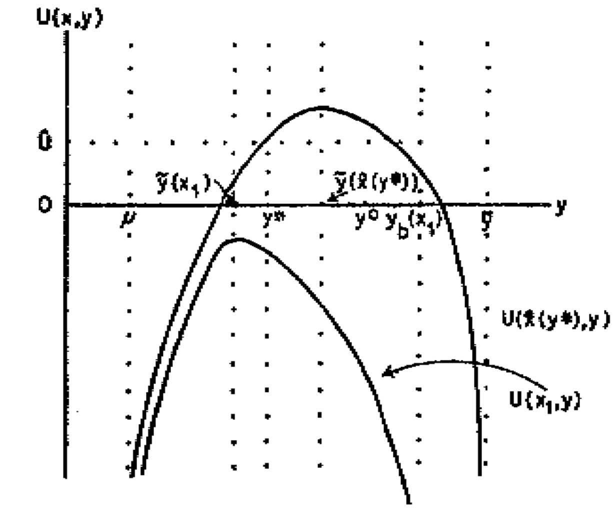

FIGURE 3.—DEPENDENCE OF THE UTILITY RATE ON THE UTILIZATION RATIO WHEN $x = \hat{x}(y^*)$ AND WHEN $x = x_1$.

Both curves are strictly concave since $U_{yy} < 0$. (It can now be pointed out that $r''(y) > 0$ is unnecessarily strong for U_{yy} everywhere, let alone for $U_{yy} < 0$ in the neighbourhood of $\bar{y}$ as consideration of Figure 2 will show. One can simply postulate $U_{yy} < 0$ noting that this prohibits $r''(y)$ from being excessively negative.) Both curves reach a peak—the static optimum—left of y^0 since $x < \hat{x}(y^0)$ in both cases. The top curve reaches a peak to the right of y^* because at $y = y^*$ there is full liquidity, so $\varphi_1 = 0$ (right-hand as well as left-hand derivative), while $\varphi_2 > 0$ because $y^0 > y^*$, so that $U_y[\hat{x}(y^*), y^*] > 0$, i.e., the curve must still be rising at y^*. For purposes of illustration it was assumed that $y_b[\hat{x}(y^*)] \geqslant \tilde{y}$ so that the right-hand asymptote is $\tilde{y}$. The lower curve, corresponding to a much smaller x, has the same shape but reaches a peak, $\bar{y}(x_1)$, to the left of y^*. This is because, in the case illustrated (if x is very small), the marginal gain from higher utilization at $y = y^* < y^0$ is not worth the concomitant increase of interest rate because the interest rate is already so high in this case. [It should be remarked that the portion of the solution (discussed later) which can be regarded as "deflationist" is not in any way dependent upon the fact that, for sufficiently small x, $\hat{y}(x) < y^*$; deflation (or at least $y < y^*$) can be optimal even for x much higher than the aforementioned value, i.e., even when the static optimum is always above y^*.] Looking at the right-hand asymptote, this reflects the fact that for sufficiently small x, $y_b(x) < \bar{y}$. I have assumed for definiteness that $y_b(x_1) > y^*$, but the reverse inequality is certainly possible. Note finally, for completeness, that $y_b(x)$ approaches μ asymptotically as x falls and approaches $x_b(\mu)$.

Figure 4 illustrates the dependence of the utility rate on x for two given values of y: first, $y = y^*$ so that there will be full liquidity at $x \geqslant \hat{x}(y^*)$; second, $y = y_1 < y^*$. Both curves are, loosely speaking, reverse images of the curve (not drawn but fully discussed) of φ against i since, with y fixed, every one point increase of x is a one point decrease of i. Both curves are concave, strictly concave outside the full-liquidity range. Consider the former curve. It is assumed for illustration only that $\hat{x}(y^*) > 0$, meaning that, in equilibrium, deflation is necessary for full liquidity. As x is decreased—the expected *inflation* rate increased—

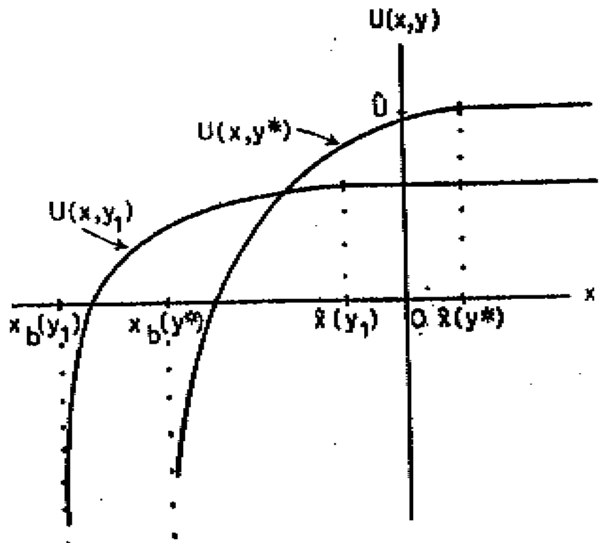

FIGURE 4.—DEPENDENCE OF THE UTILITY RATE ON EXPECTED RATE WHEN $y = y^*$ AND WHEN $y = y_1$.

the money rate in interest is increased (at a constant rate) so the rate of utility falls—at an increasing rate by virtue of the strict concavity of φ in

i. As x approaches $x_b(y^*)$, so that i approaches the barter point, the rate of utility goes to minus infinity. The other curve, corresponding to a smaller y, has the same shape. However, because y is smaller in this case and therefore i is smaller for every x, the critical rate x_b which drives the system into barter is algebraically smaller than in the previous case; i.e., a higher expected *inflation* rate is consistent with $i < i_b$ when y is smaller. Similarly, a smaller algebraic deflation rate, namely $\hat{x}(y_1)$, is needed for full liquidity. Note that since $y_1 < y^* < y^0$, full liquidity ($i \leqslant \hat{i}$) in this case gives a lower rate of utility than does full liquidity in the previous case where $y = y^*$. While it is of no significance, these considerations imply that the two curves cross: at algebraically very small x, $y^* > y_1 > \hat{y}(x)$ so that $y^* > y_1$ actually reduces the rate of utility in that range of x.

Before (8) is utilized, some defence of it and consideration of alternatives is in order. Consider the poor German worker of the early 1920s. He was not in the market for equities so that for him the real interest rate was zero; or, rather, for him the real interest rate was only the convenience yield of holding a stock of consumer durables (cigarettes, bottled beer, etc.) which we might regard as becoming rapidly negligible as this stock is increased. It could be argued that for such people the appropriate utility-rate function is better described by $U = \psi(-x, y)$ on the ground that the opportunity cost of holding money is simply the expected rate of inflation. If we make assumptions like $\psi_{21} < 0$ in the spirit of (7) we can still arrive at (8). There is little to be gained except simplicity from this approach at the cost of neglecting altogether the role of the real rate of interest for those people who participate in the capital market and who own a substantial amount of the wealth.

Another issue is my omission of the *actual* inflation rate from (7). Observe that, by virtue of (3) which makes the inflation rate a function of x and y, the utility rate must ultimately depend on x and y, as in (8). We could write

$$U = \psi(\dot{p}/p, i, y) = \psi[f(y) - x, r(y) - x, y]$$

and still obtain some version of (8). The issue therefore revolves only around the shape of the function in (8).

I have already given full weight to the loss of utility arising from a discrepancy between the actual and expected rates of inflation. It is in large part this discrepancy that motivates opposition to inflation. It is not really inflation *per se* that many economists oppose but rather an unexpectedly high rate of inflation. Nevertheless it might be argued that it is of no consolation to fixed-income groups to guess correctly the current rate of inflation if they did not *anticipate* when they contracted their fixed money incomes the bulk of the inflation that has occurred in the intervening time!

On one interpretation, this is a distributional argument: the real incomes or real wealth of widows and orphans on previously contracted fixed incomes will be eroded to socially undesirable levels by inflation. My grounds for omitting the actual inflation rate, from this point of view, must be that the government has other means than the depressing of the utilization ratio to rectify tolerably the distribution of income.[1]

To the extent that appropriate redistribution efforts still leave such groups too poor, there is certainly a case for introducing the actual rate of inflation into the utility-rate function, ψ. But it is enormously difficult to introduce it appropriately. For if the actual and expected inflation rates should be equal for a long time then the actual rate of inflation deserves less and less weight over time; for eventually the inflation will have become a *fully anticipated* one. Thus an appropriate utility-rate function must be a non-stationary function. No simple possibilities satisfy me. But I wish to point out that since the optimal path in my model produces asymptotically a steady rate of algebraic inflation, hence an asymptotically anticipated inflation, and since the rate of an anticipated inflation makes no difference distributionally (apart from its liquidity effect already recognized), the asymptotic properties of the solution here are immune to criticism from this point of view.

The actual inflation rate has another influence which, it could be argued, is time-independent and hence persisting for all time. This is the nuisance cost of adjusting price lists up or down. If the rate of inflation is 20 per cent. or —20 per cent. per annum, every firm in every industry will have to revise its price lists very frequently, which again has its leisure or production costs. This suggests giving the actual rate of inflation a weak role in the utility-rate function. $\psi(\ \ )$ can be made a dome-shaped function of $\dot{p}/p$. The concavity of U in y would be threatened a little—precautions would be needed to insure that $U_{yy} < 0$ everywhere—but not much of (8) would be lost. The main difference is that instead of having a U maximum in the x plane for all $x \geqslant \bar{x}(y)$ we would have a unique, non-flat peak in Figure 4, since too high an expected rate of deflation would cause too high an actual deflation rate from the point of view of price lists. I shall mention in the next section an instance where it would be useful to introduce such a modification.[2]

My greatest reservations centre on the stationarity of the utility-rate function in (7). Suppose that $\lambda = 0$. Due to virtual golden-age growth, aggregate consumption and leisure will be growing at rate γ, like population, at any constant utilization ratio. Since the "pie" is getting bigger over time, should not U be made to depend upon t ? Fortunately, however, *per capita* consumption and *per capita* leisure, which depend only on l and y—will be constant so that the use of a stationary utility-

[1] On another view the government has a moral obligation to valididate the expectations held by groups who have contracted for fixed incomes (whether or not they are poor), even to the extent that if inflation has occurred recently the government now owes these groups a little deflation. The government of my model treats such obligations as "bygones", worrying only about the consequences of current deceptions, not past ones.

[2] This price-list consideration perhaps ought also to enter in a complicated, non-stationary way since a high, *steady* rate of inflation might eventually call forth institutional changes in the nature of money or perhaps even some system of "compounded prices".

rate function is not wholly unreasonable. The real issue here is "discounting".

More serious difficulties arise when $\lambda > 0$. Then a constant l and y imply exponentially growing consumption per head and constant leisure per head (by virtue of the labour supply function's properties). In this case it does seem a little strange that time should not appear as an argument of the utility-rate function. But I believe that examples of underlying utility functions could be found such that time would not appear in the derived utility-rate function φ in (7).

I shall however allow the rate of utility to be "discounted" at a non-negative rate in the usual multiplicative way. No solution to our problem in its present formulation will exist if there is negative discounting.

In deciding which of two (x,y) paths to take—actually $x(t)$ alone suffices to describe a path—the Fisc is postulated to compare the integrals of the possibly discounted rates of utility produced by the two paths. Hence the "social utility", W, of a path (x,y) is given by

$$(9)\quad W = \int_0^\infty e^{-\delta t} U(x,y)\, dt, \qquad \delta \geqslant 0,$$

where t is time, $e^{-\delta t}$ is the discount factor applied to the rate of utility t years hence, and δ is the rate of utility discount. (It is understood in (9) that $x = x(t)$, $y = y(t)$.) The case $\delta = 0$ will receive special consideration in a moment.

The optimization problem of the Fisc can now be stated as: maximize (9) subject to (5) and (6). The "optimal policy" is the function $y = y(x)$ which gives the greatest feasible W. Given $x(0) = x_0$, there is an optimal path $x = x(t)$ which describes the state of the system at each time. From this information one can also derive $y = y(t)$, since $\dot{x}(t)$ gives $y(t)$ by (5).

In the case $\delta = 0$, there may be many feasible paths which cause the integral in (9) to diverge to infinity, which give infinite W; intuitively, it is unreasonable to regard all of these paths as "optimal" so that a different criterion of preferences and of optimality is wanted in this case. Such a criterion will be described briefly in the next section, which also gives the solution to the zero-discount case. (Nevertheless the above formulation of the mathematics of optimization is essentially correct.) The subsequent section gives the solution to the case of a positive utility discount rate.

II. Optimal Policy when no Utility Discounting

The optimality criterion now widely used by economists to deal with no-discount, infinite-horizon problems of this sort has been called the "over-taking principle". A path $[x_1(t), y_1(t)]$ is said to be preferred or indifferent to another path $[x_2(t), y_2(t)]$ if and only if one can find a time T^0 sufficiently large that, for all $T > T^0$,

$$\int_0^T U(x_1, y_1)\, dt \geqslant \int_0^T U(x_2, y_2)\, dt.$$

The former path is preferred because it eventually "overtakes" the latter path. A feasible path is said to be *optimal* if it is preferred or indifferent to all other feasible paths. If one then obtains a solution to the maximization problem now to be described, this solution is the optimum in this sense.[1]

The above optimality criterion justifies the use of a device first employed by Ramsey in his analysis of the somewhat analogous problem of optimal saving over time: choose the units in which the utility rate is measured in such a way that $\hat{U} = 0$, i.e., $U[\hat{x}(y^*), y^*] = 0$. This is merely a linear transformation of the function U that will not affect the preference orderings implied by the integral comparisons just described. Now go ahead with the problem

$$\text{(10)}\quad \max_{y} W = \int_0^{\infty} U(x,y)\,dt, \qquad \hat{U} = 0,$$
$$\text{subject to } \dot{x} = G(y), \qquad x(0) = x_0.$$

The divergence problem cannot now arise. This is not to say, however, that an optimal policy will exist for all x_0.

Readers familiar with the Ramsey problem will recognize (10) as rather like the "optimal saving" problem. There x is "capital" and y is "consumption".[2] There is a zero-interest capital-saturation level in Ramsey that is analogous to our liquidity satiation level, $\hat{x}(y)$; his income—the maximum consumption subject to constant capital—is analogous to our y^*. His solution was the following. If initial capital is short of capital saturation, consume less than income, driving capital up to the saturation level; if initial capital exceeds the saturation level, consume more than income, driving capital down to the saturation level; if initial capital equals the capital-saturation level, stay there by consuming all capital-saturation income. Thus capital either equals for all time or approaches asymptotically and monotonically the capital-saturation level while consumption either equals or approaches asymptotically (and monotonically) the capital-saturation level.

The solution to the problem here is similar in part. If $x_0 < \hat{x}(y^*)$ it is optimal to make $y < y^*$ for all t, causing x to rise and approach $\hat{x}(y^*)$ asymptotically, while y approaches y^* asymptotically and monotonically. In other words, if the economy "inherits" an initially expected algebraic deflation rate that is insufficient for full liquidity when the utilization ratio is at its equilibrium value, then, for an optimum, the Fisc must engineer under-utilization for all time so as to cause a gradual, asymptotic movement of the expected deflation rate up to the level consistent with full liquidity and equilibirum utilization; in the limit, as time

[1] See, for example, "The Ramsey Problem and the Golden Rule of Accumulation" in E. S. Phelps, *Golden Rules of Economic Growth*, New York, 1966, and the references cited there.

[2] Some differences are that his utility rate was independent of capital; his investment-consumption relation, G, depended upon capital; utility was everywhere increasing in consumption; and $G'(y) = -1$ in his case.

increases, under-utilization vanishes and a full-liquidity equilibrium is realized.

If $x_0 = \hat{x}(y^*)$ then $y = y^*$ is optimal for all t, and therefore $x = \hat{x}(y^*)$ for all t. Should the economy inherit the minimum expected deflation rate consistent with full liquidity at equilibrium utilization, then equilibrium utilization with full liquidity is optimal for all time. The case $x_0 > \hat{x}(y^*)$ will be discussed later.

What will be remarkable to those steeped in the statical approach is that, when $x_0 \leq \hat{x}(y^*)$, over-utilization is not optimal whether or not x is large enough to make $\bar{y}(x) > y^*$. Further it can be shown that optimal y is always smaller than $\bar{y}$ even when $\bar{y} < y^*$.

Analogous to the Ramsey-Keynes equation that gives optimal consumption as a function of capital is the following equation that describes optimal utilization as a function of the current expected deflation rate.[1]

$$\text{(11)}\quad U(x,y) + G(y)\frac{U_y(x,y)}{-G'(y)} = 0.$$

For purposes of diagrammatics it is helpful to write $U = V(x,\dot{x}) = V[x, G(y)]$, which we may do since $G(y)$ is monotone decreasing in y, and then to express (11) in the form

$$\text{(12)}\quad V(x,G) - G\,V_G(x,G) = 0$$

where
$$V_G = \frac{U_y(x,y)}{G'(y)}, \qquad V_x = U_x,$$
$$V_{GG} = \frac{U_{yy}G' - G''U_y}{G'G'G'}, \qquad V_{Gx} = \frac{U_{yx}}{G'(y)}.$$

If we think of $\dot{x} = G(y)$ as "investment", then (12) says that the optimal policy equates the rate of utility to investment multiplied by the (negative) marginal utility of investment, V_G; this is essentially the Ramsey-Keynes rule.

From the information above on derivatives we see that V *increases* as G is increased [i.e., as y is decreased from $\bar{y}$ or $y_h(x)$, whichever is smaller] up to $G(\bar{y})$ whereupon V then decreases, going to minus infinity as G approaches $G(\mu)$. Only this latter decreasing region, where $V_G < 0$ or $U_y > 0$, is of relevance; in that region, $V_{GG} < 0$ unambiguously.

In Figure 5 the solid curve depicts the possibly realistic case of x_0 great enough that $\bar{y}(x_0) > y^*$, so that $G[\bar{y}(x_0)] < 0$, but not great enough for full liquidity when $y = y^*$, i.e., $x_0 < \hat{x}(y^*)$. Thus the solid utility curve, for $x = x_0$, has a peak left of the origin but it passes under the origin, since $U(x_0,y^*) < \hat{U} = 0$. The tangency point, at (V_0, G_0), shows the optimal initial $G(y)$ and hence the optimal y. Since optimal $G(y) > 0$ (i.e., $y < y^*$), x will be increasing and the V curve will therefore shift up and possibly to the left; as this process occurs, the tangency

[1] For a simple derivation, in which the differentiability necessary for the Euler condition is not assumed, see R. E. Bellman, *Dynamic Programming*, Princeton, 1956, pp. 249-50.

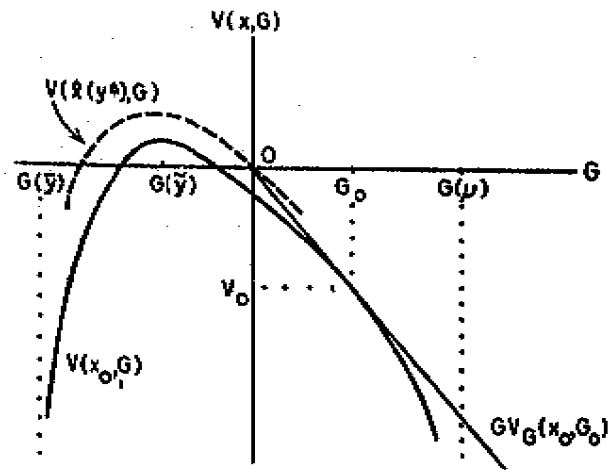

FIGURE 5.—THE NO-DISCOUNT UTILIZATION OPTIMUM WHEN $x_0 < \hat{x}(y^*)$.

point approaches the origin, so that $y = y^*$ and $x = \hat{x}(y^*)$ in the limit.[1] The dashed curve represents the asymptotic location of the V curve. Just as equilibrium utilization is approached only asymptotically, it can be shown that full liquidity ($i \leq \bar{i}$) is approached only asymptotically. (This follows from $r'(y) > 0$ and the results that $U_y > 0$ along the optimal path.)

The case $x_0 = \hat{x}(y^*)$ is now obvious. Here we are in long-run equilibrium to begin with, as shown by the dashed V curve in Figure 5. The tangency point occurs at the origin so $y = y^*$ is optimal initially; this means that the equality $x(t) = \hat{x}(y^*)$ continues so that $y = y^*$ continues to be optimal for all t.

Consider now the case $x_0 > \hat{x}(y^*)$. Since there cannot be *more* than full liquidity when $y = y^*$, i.e., $U(x_0,y^*) = \bar{U}$ even when $x_0 > \hat{x}(y^*)$, the tangency point continues to be at the origin. Yet the implied policy $y(t) = y^*$, $x(t) = x_0 > \hat{x}(y^*)$ for all t *cannot be optimal*. For there is a "surplus" of expected deflation here; i.e., $i < \bar{i}$ when $y = y^*$. Since V reaches a peak to the left of y^*, there are clearly policies of at least temporary over-utilization ($y > y^*$) which will permit $U > \bar{U}$ for at

[1] The reader may have noticed a second tangency point with $G < 0$. Pursuit of that policy would lead asymptotically to $y = y^*$ with $x = \tilde{x}$ where $\tilde{y}(\tilde{x}) = y^*$; since $\tilde{x} < \hat{x}(y^*)$, such a policy must cause W to diverge to minus infinity so that it cannot be optimal.

least a while and yet allow $U = \bar{U}$ forever after; this is because $x = \hat{x}(y^*) < x_0$ is sufficient for $U(x,y^*) = \bar{U}$. In other words, there is room for a "binge" of at least temporary over-utilization while all the time enjoying full liquidity and while never driving x below $\hat{x}(y^*)$.

But it cannot be concluded that over-utilization is optimal when $x_0 > \hat{x}(y^*)$. For no such temporary or even asymptotically vanishing binge of over-utilization can satisfy (12), which is a necessary condition for an optimum; in terms of Figure 5, there is no way that such a policy can satisfy the necessary tangency condition.

Since neither $y > y^*$, $y = y^*$ nor $y < y^*$ is optimal, the inescapable conclusion is that there exists no optimum in this case. An intuitive explanation is the following. For every binge that you specify which makes $x(t)$ approach $\hat{x}(y^*)$ (as y approaches y^*), I can, by virtue of the strict concavity of the V curve, specify another binge that makes x approach $\hat{x}(y^*)$ more slowly which will be even better. There is no "best binge" (or even set of "best binges") just as there is no number closest to unity yet not equal to it. Hence there is no path preferred or indifferent to all other feasible paths.

There are at least four avenues of escape from this disconcerting situation. Let us first ask, how did Ramsey avoid it? He could avoid it (actually he never recognized it) by postulating that the net marginal product of capital became negative beyond the capital saturation point so that there was an immediate and positive loss from having too much capital. (This is fair enough if capital depreciates even in storage.) In our model there is no immediate loss from having "too high" an expected deflation rate; $i < \bar{i}$ is as good as $i = \bar{i}$. To introduce a loss we need to suppose that U in (8) is strictly concave in x, reaching a peak and falling off thereafter. As mentioned earlier, this postulate could be justified by the price-list consideration that it is a nuisance to have to reduce prices with great frequency. (But a previous footnote indicates my uneasiness with this consideration.) Alternatively one could make assumptions leading to $G_x(x,y) < 0$, as is done in the preliminary version of this paper.

Another avenue of escape is the introduction of a positive utility discount, as I have done in the next section. Then there will be a "best binge" so there will be an optimum for all x_0 (in the admissible range).

A third avenue is to employ a finite-time horizon. Then any binge must come to an end at the end of some given number of years. There will be a "best binge" and an optimum will always exist. The unpublished version of this paper contains such a model.

The fourth avenue of escape is to postulate that $y^0 = y^*$ so that $\bar{y}(x) \leq y^*$ for all x and therefore the V peak cannot occur to the left of the origin. I find this unsatisfactory although some readers may not. The reader can now work out this case using a diagram like Figure 5. If $x_0 > \hat{x}(y^*)$, under-utilization is optimal as before; if $x_0 \geq \hat{x}(y^*)$, equilibrium utilization is optimal. Anyone who wants to go as far as postulating $y^0 < y^*$ will encounter problems of the non-existence of an optimum.

Some of the qualitative results of this section may be expressed by the following "policy function" derived from (12):

$$y = y(x), \qquad x \leq \hat{x}(y^*), \tag{13}$$

$$\text{where } y'(x) \left\{ \gtreqless \right\} 0 \text{ as } x \left\{ \lesseqgtr \right\} \hat{x}(y^*),$$

$$y(x) \begin{cases} = y^* \text{ if } x = \hat{x}(y^*), \\ < y^* \text{ if } x < \hat{x}(y^*), \end{cases}$$

$$\lim_{x \to x\hat{x}(\mu)} y(x) = \mu.$$

Let us turn now to the mathematically more congenial case of a positive discount.

III. Optimal Policy When Positive Utility Discounting

Our problem now is

$$\text{Max}_{y}\ W = \int_0^\infty e^{-\delta t}\, U(x,y)\, dt, \qquad \delta > 0, \tag{14}$$

$$\text{subject to } \dot{x} = G(y), \quad x(0) = x_0.$$

A mathematical analysis, in which (14) is a special case, is contained in the preliminary version of this paper. I shall describe the solution here.

The optimal path of the variable $x(t)$ either coincides with or monotonically approaches (from every x_0) a "long-run equilibrium" value, x^*, which is uniquely determined by

$$\delta = \frac{-U_x(x^*, y^*)\ G'(y^*)}{U_y(x^*, y^*)}. \tag{15}$$

It is easy to see from (15), the inequality $G'(y) < 0$ and the observation that an optimal path would never make $U_y(x,y) < 0$, that $U_x(x^*,y^*)>0$. This and (8) yield the result that $x^* < \hat{x}(y^*)$. Thus, in the long run, there will be less than full liquidity when there is positive discounting of future utility rates. This is because the current gain from high utilization always offsets the *discounted* future loss due to a short fall from full liquidity.

If $x_0 < x^*$, so that the expected deflation rate is below its long-run optimal value, then, to drive $x(t)$ monotonically toward x^* we require $y < y^*$, i.e., under-utilization; $y(t)$ will approach y^* only asymptotically as $x(t)$ approaches x^*. If $x_0 = x^*$, then $y = y^*$ is optimal for all t. If $x_0 > x^*$, then, to drive $x(t)$ monotonically toward x^* we require $y > y^*$, i.e., over-utilization; but, again, $y(t)$ will approach y^* asymptotically. (It does not appear that the path $y(t)$ is necessarily monotonic but this is of little importance.)

This last result—the optimality of over-utilization in some circumstances—is of considerable interest. The previous section laid a possible foundation for a "deflationist" policy when the initially expected deflation rate was insufficient for full liquidity with equilibrium utilization; more precisely, under-utilization was optimal in that circumstance so that the actual rate of *inflation* resulting would be less than the expected rate, though it need not be negative initially [or even asymptotically if $\hat{x}(y^*) \leq 0$]. Moreover, an "inflationist" policy of over-utilization, though it might be better than any under-utilization policy, was never optimal for there could never exist an over-utilization optimum. We see here that, when there is a positive utility discount, over-utilization will be optimal when $x_0 > x^*$; since $x^* < \hat{x}(y^*)$, this embraces the case $x_0 = \hat{x}(y^*)$, i.e., the case in which there would be full liquidity at equilibrium utilization.

The greater is the utility discount rate, the smaller algebraically will be the equilibrium deflation rate. Differentiation of (15) yields

$$\frac{dx^*}{d\delta} = \frac{[U_y(x^*,y^*)]^2}{[U_{yx}(x^*,y^*)U_x(x^*,y^*) - U_{xx}(x^*,y^*)U_y(x^*,y^*)]G'(y^*)} < 0, \tag{16}$$

since the denominator is unambiguously negative for all $x^* < \hat{x}(y^*)$, hence for $\delta > 0$. This indicates that, given some x_0, we are more likely to find over-utilization initially optimal ($x_0 > x^*$) the larger is the utility discount rate.

Nevertheless one cannot, by choosing sufficiently large δ, make x^* arbitrarily small (algebraically), not even as small as $x_0(y^*)$. It is the inequality $\bar{y}(x^*) > y^*$ that lies behind the optimality of $y > y^*$ when $x_0 > x^*$. It can be shown that x^* cannot be made larger than $\bar{x}(y^*)$, where $\bar{x}$ is defined by $\bar{y}(x) = y^*$; for as δ goes to infinity, the derivative $U_y(x^*,y^*)$ in (15) goes to zero (while $U_x(x^*,y^*)$ stays finite), indicating that x^* approaches the value such that $U_y(x,y^*) = 0$, hence approaches the value $\bar{x}(y^*)$.

The value $\bar{x}(y^*)$ is precisely the level of x to which the myopic, statical approach would drive $x(t)$. That approach, which maximizes the current rate of utility at each time, leads to a policy $y = \bar{y}(x)$; under that policy, equilibrium is realized only when (asymptotically) $x = \bar{x}(y^*)$ so that $\bar{y}(x) = y^*$. Thus the statical approach and the case of an infinitely high discount rate lead to the same equilibrium value of x. Indeed, it can be shown that infinite utility discounting makes $U_y(y,x) = 0$ always, which means $y = \bar{y}(x)$, so that the statical approach and infinitely heavy discounting lead to identical policies throughout time.

But optimal behaviour in the limit as δ goes to infinity is of little interest. Given any (finite) value of δ, the dynamic approach yields different results from the statical policy $y = \bar{y}(x)$. First, since $U_y(x,y) > 0$ along any dynamically optimal path, the optimal $y < \bar{y}$ for all x. Second, and this needs emphasis, even if x_0 is such that $\bar{y}(x_0) > y^*$, so that myopic maximization of the initial rate of utility would call for $y > y^*$, the truly optimal $y < y^*$ if (and only if) $x_0 < x^*$. Thus, if the currently expected rate of inflation is 2 per cent. while the long-run equilibrium (asymptotically optimal) expected inflation rate is less, say 1 per cent., then under-utilization is optimal whether or not the current utility-rate curve peaks to the right of y^*. This theme is essentially a repetition of a theme of the previous section: a dynamical approach can lead to an optimal policy that is qualitatively different from that of a myopic, statical approach. In particular, a "deflationist" policy of under-utilization (and hence a rise of x over time) may be optimal even when myopic maximization of the current rate of utility calls for over-utilization (and hence a fall of x over time).

The above results may be summarized in a qualitative way as follows.

$$y = \bar{y}(x)$$

$$\text{(17)}\quad \text{where } \bar{y}(x) \begin{cases} > y^* \text{ if } x_o > x^*, \\ = y^* \text{ if } x_o = x^*, \\ < y^* \text{ if } x_o < x^*, \end{cases}$$

$$\lim_{x \to x\delta(\mu)} \bar{y}(x) = \mu, \quad \bar{y}(x) < \tilde{y}(x) \text{ for all } x,$$

$$\text{with } \bar{x} < x^*(\delta) < \hat{x}(y^*) \text{ for all } \delta > 0, \quad x^{*\prime}(\delta) < 0.$$

Once again we may ask, what if $y^0 = y^*$? Then $\tilde{y}(x) \leq y^*$ for all x. In this event, $y < y^*$ when $x_o < x^*$ as above. And if $x_o \geq x^*$, then $y = y^*$; hence there is no over-utilization, because there is no gain to be had in the present (from over-utilization) that is worth a discounted future loss (from a reduction of future liquidity).

IV. Concluding Remarks

The principal theme here has been that, within the context of the above model, a tight fiscal policy producing "under-utilization", and hence producing an actual algebraic inflation rate that is smaller than the currently expected inflation rate, is optimal if and only if the currently expected inflation rate exceeds the asymptotically optimal inflation rate. The latter is determined by liquidity considerations and by social time preference (the utility discount rate), not by the strength of preferences for high or low utilization (at a given rate of interest). If the utility discount rate is zero, the asymptotically optimal inflation rate is simply the maximum expected inflation rate consistent with full liquidity (at equilibrium utilization). If there is positive discounting of future utility rates, the long-run inflation rate exceeds the full-liquidity rate and is greater the larger is the discount rate. From this point of view, therefore, what characterizes the advocates of a "high-pressure" policy of over-utilization is their implicit adoption of a large utility discount. In favouring high utilization today at the cost of high inflation in the eventual future equilibrium, they reveal high "time preference".

Dynamical models of this sort are a methodological step forward from the statical approach to optimal aggregate demand discussed at the outset of this article. But it would be premature to base policy on the particular model employed here. Among a host of needed extensions, the following stand out. Inflation should be made to depend upon the change of utilization, as well as the level. Investment should be made endogenous and possibly even optimized simultaneously with aggregate demand. And where it is appropriate to assume fixed or only occasionally adjustable exchange rates, balance-of-payments considerations

should be introduced; from this viewpoint, the model's greatest relevance may be for a nation's optimal objectives in the international co-ordination of aggregate demand and price trends among countries.

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